TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 304, Number 1, November 1987

THE MORAVA K-THEORIES OF SOME CLASSIFYING SPACES

NICHOLAS J. KUHN

ABSTRACT. Let P be a finite abelian p-group with classifying space BP. We compute, in representation theoretic terms, the Morava K-theories of the stable wedge summands of BP. In particular, we obtain a simple, and purely group theoretic, description of the rank of $K(s)^*(BG)$ for any finite group G with an abelian p-Sylow subgroup. A minimal amount of topology quickly reduces the problem to an algebraic one of analyzing truncated polynomial algebras as modular representations of the semigroup $M_n(\mathbf{Z}/p)$.

1. Main results. For a fixed prime p, there exists a sequence of cohomology theories $K(1)^*$, $K(2)^*, \ldots$ with $K(s)^*$ periodic with period $2(p^s - 1)$. These are the Morava K-theories [**13**], and generalize ordinary complex K-theory in the sense that $K(1)^*$ is one of the $(p-1)$ isomorphic summands of $K^*(\ ;\mathbf{Z}/p)$. Recently it has been becoming clear that they play a central role in homotopy theory—see, for example, the work of M. Hopkins and J. Smith [**6**]. $K(s)^*$ has further computational virtues: $K(s)^*(X)$ is always a free module over the coefficient ring $K(s)^*$, and there is a Kunneth isomorphism.

A major outstanding problem is to find good models for the spaces representing these theories. Related to this is the question of finding, for finite groups G, a group theoretic description for the rings $K(s)^*(BG)$, analogous to Atiyah's isomorphism [**2**]:

$$\hat{R}(G) \simeq K(BG).$$

The author, together with J. Harris, recently analyzed stable wedge decompositions of classifying spaces of abelian p-groups [**5**]. In this paper we compute, in representation theoretic terms, the Morava K-theories of the resulting wedge summands. As a consequence, we obtain a very simple, and purely group theoretic, description of the rank of $K(s)^*(BG)$ for any finite group G with an abelian p-Sylow subgroup.

To state our results, we establish some notation. It is convenient to let $K(0)^*$ be rational cohomology. We let $K(s)^*(X)$ always denote the reduced sth Morava K-theory of X. If X is a space or spectrum, we let $k_s(X)$ equal the rank of $K(s)^*(X)$ as a $K(s)^*$-module, and, by abuse of notation, we let $k_s(G) = k_s(BG_+)$ for a finite group G. (BG_+ is the union of BG with a disjoint basepoint.)

Received by the editors May 27, 1986.
1980 *Mathematics Subject Classification*. Primary 55N22; Secondary 20G05.
Research partially supported by the Sloan Foundation and the NSF.

If P is a finite p-group, let $\mathbf{Z}/p[\mathrm{End}(P)]$ be the ring with basis the semigroup $\mathrm{End}(P)$. $\mathrm{End}(P)$ acts on BP and thus on BP_+. It is easy to show that an idempotent e in $\mathbf{Z}/p[\mathrm{End}(P)]$ yields a stable wedge summand of BP_+, eBP_+, such that, e.g., $H_*(eBP_+; \mathbf{Z}/p) = e_*H_*(BP_+; \mathbf{Z}/p)$. In [5], we show that, if P is abelian, *every* stable wedge summand of BP_+ is homotopic to one of this form.

Note that $\mathrm{End}(P)$ acts diagonally on the set P^s so that, letting $\mathbf{Z}/p[P^s]$ denote the $\mathbf{Z}/p$-vector space with basis P^s, $\mathbf{Z}/p[P^s]$ is a $\mathbf{Z}/p[\mathrm{End}(P)]$-module.

THEOREM 1.1. *Let P be a finite abelian p-group and let e be an idempotent in $\mathbf{Z}/p[\mathrm{End}(P)]$. Then*

$$k_s(eBP_+) = \dim e\mathbf{Z}/p[P^s].$$

Now let G be a finite group with p-Sylow subgroup P. By transfer arguments, BG_+ is a stable wedge summand of BP_+, localized at p. In our situation we can be more explicit. Let $W = N_G(P)/C_G(P)$ and let $e_W = |W|^{-1}\Sigma_{w \in W} w$ in $\mathbf{Z}/p[\mathrm{End}(P)]$. Then, if P is abelian, $BG_+ \simeq e_W BP_+$ [5]. Furthermore, $e_W\mathbf{Z}/p[P^s] = \mathbf{Z}/p[P^s]^W$, a vector space with a basis corresponding to W-orbits in P^s. Theorem 1.1 thus implies

THEOREM 1.2. *If G is a finite group with an abelian p-Sylow subgroup P, and $W = N_G(P)/C_G(P)$, then $k_s(G) = |P^s/W|$.*

REMARKS 1.3. (i) Doug Ravenel has noted that $k_s(G)$ is finite for all finite groups G [12].

(ii) Using Atiyah's theorem and some representation theory, one can show [7] that for any finite group G,

$$k_1(G) = \text{number of conjugacy classes of } p\text{-elements in } G.$$

It is an amusing exercise using the Sylow theorems to check that this result is compatible with Theorem 1.2 above.

When $P = (\mathbf{Z}/p)^n$ one can identify $\mathrm{End}(P)$ with the matrix ring $M_n(\mathbf{Z}/p)$, P^s with the set $M_{n,s}(\mathbf{Z}/p)$ of $n \times s$ matrices over $\mathbf{Z}/p$, and the action of $\mathrm{End}(P)$ on P^s with matrix multiplication. In particular, letting $s = n$ in Theorem 1.1 implies the following.

COROLLARY 1.4. *Let M be an irreducible right $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-module, let P_M be the associated principal indecomposable module (i.e., its projective cover), and let X_M be the associated wedge summand of $B(\mathbf{Z}/p)^n_+$. Then*

$$k_n(X_M) = \dim P_M.$$

Thus, no summand of $B(\mathbf{Z}/p)^n$ is $K(n)$-acyclic.

At the other extreme, we show

THEOREM 1.5. *Of the $p^n - 1$ distinct indecomposable spectra that appear as wedge summands of $B(\mathbf{Z}/p)^n$, exactly $(p-1)n$ are not acyclic in K-theory. For each such summand X, $k_1(X) = 1$.*

Further analysis of $\mathbf{Z}/p[M_{n,s}(\mathbf{Z}/p)]$ as a $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-module yields the next results.

THEOREM 1.6. (1) *For each pair (n, s), there exists a linear function $\alpha_{n,s}\colon \mathbf{Z}^{n+1} \to \mathbf{Z}$ such that, if X is any wedge summand of $B(\mathbf{Z}/p)^n_+$,*

$$k_s(X) = \alpha_{n,s}(k_0(X), \ldots, k_n(X)).$$

Thus $k_s(X)$ is determined for all s by $k_0(X), \ldots, k_n(X)$.

(2) *If P is a finite abelian p-group and X is a wedge summand of BP, then $\sum_{s=0}^{\infty} k_s(X)t^s$ is a rational function with poles contained in the set $\{p^{-s} \mid s = 0, 1, 2 \ldots\}$.*

(3) *If X is as in (2), the sequence $k_1(X)$, $k_2(X)$, $k_3(X), \ldots$ converges, in the p-adic topology, to an element of $\mathbf{Z}_{(p)}$.*

We note that the linear functions $\alpha_{n,s}$ will be made more explicit in the course of the proof.

Finally we note that J. F. Adams, J. Gunawardena and H. Miller have shown [**1**]:

$$\mathbf{Z}/p[M_{n,s}(\mathbf{Z}/p)] \simeq \operatorname{Hom}_A(H^*(B(\mathbf{Z}/p)^s_+), H^*(B(\mathbf{Z}/p)^n_+))$$

where A is the mod p Steenrod algebra. Combined with our observations, this yields the amusing corollary:

COROLLARY 1.7. *If X is a stable wedge summand of $B(\mathbf{Z}/p)^n$, for some n, then*

$$k_s(X) = \dim \operatorname{Hom}_A(H^*(B(\mathbf{Z}/p)^s_+), H^*(X)).$$

The organization of the paper is as follows. Theorem 1.1 is proved in §3 after we discuss $K(s)^*(BP_+)$ in §2. Theorem 1.5 is proved in §4, Theorem 1.6 in §5. §6 contains some explicit calculations, e.g. $k_s(\mathrm{GL}_2(\mathbf{F}_q))$ where $q = p^d$, and $k_s(L(n))$ where $L(n) = \Sigma^{-n}SP^{p^n}(S)/SP^{p^{n-1}}(S)$.

We illustrate the ideas of the proof of Theorem 1.1 by sketching the argument in the case $P = (\mathbf{Z}/p)^n$.

$$K(s)^*(B\mathbf{Z}/p_+) = K(s)^*[x]/(x^{p^s}),$$

so that $K(s)^*(B(\mathbf{Z}/p)^n_+) = K(s)^*[x_1, \ldots, x_n]/(x_1^{p^s}, \ldots, x_n^{p^s})$. The right action of $M_n(\mathbf{Z}/p)$ on $K(s)^*(B(\mathbf{Z}/p)^n_+)$ is determined by the formal group law for $K^*(s)$—modulo some high-degree error terms, it is the standard action on a truncated polynomial algebra. We are left needing to show the purely algebraic result:

THEOREM 1.8. *$(\mathbf{Z}/p[M_{n,s}(\mathbf{Z}/p)])^*$ and $\mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^{p^s}, \ldots, x_n^{p^s})$ have the same irreducible composition factors as right $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-modules.*

The relevance of this result to Theorem 1.1 comes from the following elementary but handy observation, which will be used numerous times in our arguments.

LEMMA 1.9. *Let R be a finite-dimensional algebra over a field $\mathbf{F}$, and let M and N be finitely generated right R-modules. The following conditions are equivalent:*

(i) $\dim_{\mathbf{F}} Me = \dim_{\mathbf{F}} Ne$ *for all idempotents $e \in R$.*

(ii) *M and N have the same irreducible composition factors.*

Either condition is implied by

(iii) *There exist R-module filtrations of M and N such that the associated graded objects are isomorphic as R-modules.*

Thie project had its genesis during a recent visit to the University of Washington, when Doug Ravenel wrote down the numbers 1, 3, 3, 5, 2, 2 in my presence, For that, and for subsequent conversations, I give him hearty thanks. Thanks are also due to Dave Carlisle and Reg Wood for aid with the proof of Theorem 1.5.

2. $K(s)^*(BP_+)$. In this section we describe, up to filtration, the End(P)-module $K(s)^*(BP_+)$, where P is any finite abelian p-group. (See also the proof of Theorem 4.9 in [**10**].)

$M_n(\mathbf{Z}/p)$ acts on the right of the polynomial algebra $S_n = \mathbf{Z}/p[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are dual to the standard basis of $(\mathbf{Z}/p)^n$. For a fixed s, the ideal generated by the p^s powers is a submodule. We let $S_{n,s}$ denote the quotient $M_n(\mathbf{Z}/p)$ algebra $\mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^{p^s}, \ldots, x_n^{p^s})$. More generally, note that $M(n_1, \ldots, n_r)$ acts on $\otimes_{i=1}^r S_{n_i, is}$ where $n = n_1 + \cdots + n_r$ and $M(n_1, \ldots, n_r)$ is the subsemigroup of $M_n(\mathbf{Z}/p)$ consisting of matrices preserving the flag

$$(\mathbf{Z}/p)^{n_r} \subseteq (\mathbf{Z}/p)^{n_r + n_{r-1}} \subseteq \cdots \subseteq (\mathbf{Z}/p)^n.$$

(Here $(\mathbf{Z}/p)^m \subset (\mathbf{Z}/p)^n$ is the inclusion of the *last* m coordinates.)

Now suppose that $P = \prod_{i=1}^r (\mathbf{Z}/p^i)^{n_i}$. We define a "standard" action of End(P) on $\otimes_{i=1}^r S_{n_i, is}$ as follows: If Tor($P, \mathbf{Z}/p$) is identified with $(\mathbf{Z}/p)^n$ in the obvious way, then it is easily checked that Tor$(\alpha, \mathbf{Z}/p) \in M(n_1, \ldots, n_r)$ for $\alpha \in$ End(P). This defines a map of semigroups End$(P) \to M(n_1, \ldots, n_r)$, and thus a right action of End(P) on $\otimes_{i=1}^r S_{n_i, is}$.

PROPOSITION 2.1. *If* $P = \prod_{i=1}^r (\mathbf{Z}/p^i)^{n_i}$ *then, as algebras,*

$$K(s)^*(BP_+) \simeq K(s)^* \otimes \left(\bigotimes_{i=1}^r S_{n_i, is} \right),$$

and the End(P) *action is the standard one, modulo terms of higher degree.*

COROLLARY 2.2. *With P as above, and $e \in \mathbf{Z}/p[\mathrm{End}(P)]$ an idempotent,*

$$k_s(eBP_+) = \dim \left[\bigotimes_{i=1}^r S_{n_i, is} \right] e.$$

We collect the results about Morava K-theories which imply Proposition 2.1. A good reference is [**13**, §4].

We start with some generalities. If E is an MU-oriented ring spectrum, then $E^*((\mathbf{C}P^\infty)_+^n) = E^*[[x_1, \ldots, x_n]]$, where each x_i has degree 2. The multiplication $\mathbf{C}P^\infty \times \mathbf{C}P^\infty \to \mathbf{C}P^\infty$ defines a formal group law F over E^*.

The semigroup $M_n(\mathbf{Z})$ acts on $\mathbf{Z}^n$ and thus on $(\mathbf{C}P^\infty)^n = K(\mathbf{Z}^n, 2)$. The formal group law F determines the induced action on E^* cohomology: If $A = (a_{ij}) \in M_n(\mathbf{Z})$ then

$$A^*(x_j) = \sum_{i=1\,F}^{n} [a_{ij}]_F x_i.$$

Since $F(x, y) = x + y$ modulo higher-order polynomials in x and y, we conclude

LEMMA 2.3. *Let $E^*((\mathbf{C}P^\infty)^n_+) = E^*[[x_1, \ldots, x_n]]$ be filtered by degree, i.e. let $F_k E^*[[x_1, \ldots, x_n]] = \{ f(x_1, \ldots, x_n) \mid f(x, \ldots x)$ is divisible by $x^k\}$. Then this is a decreasing filtration by sub-$M_n(\mathbf{Z})$-algebras and the associated graded $M_n(\mathbf{Z})$-algebra is isomorphic to $E^* \otimes \mathbf{Z}[[x_1, \ldots, x_n]]$ with the standard action.*

Now suppose that P is a finite abelian group of rank n. Then P fits into a short exact sequence

$$0 \to \mathbf{Z}^n \to \mathbf{Z}^n \to P \to 0,$$

and any endomorphism α: $P \to P$ can be extended to a diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}^n & \to & \mathbf{Z}^n & \to & P & \to & 0 \\ & & \downarrow A_1 & & \downarrow A_0 & & \downarrow \alpha & & \\ 0 & \to & \mathbf{Z}^n & \to & \mathbf{Z}^n & \to & P & \to & 0 \end{array}$$

where A_0, $A_1 \in M_n(\mathbf{Z})$. This, in turn, induces a map of fibrations:

$$\begin{array}{ccccc} (S^1)^n & \to & BP & \overset{\delta}{\to} & (\mathbf{C}P^\infty)^n \\ \downarrow A_0 & & \downarrow \alpha & & \downarrow A_1 \\ (S^1)^n & \to & BP & \overset{\delta}{\to} & (\mathbf{C}P^\infty)^n, \end{array}$$

and Gysin sequence techniques allow for a computation of $E^*(BP)$ as an End(P)-module.

Specializing to the case of interest, we have $K(s)_* = \mathbf{Z}/p[v_s, v_s^{-1}]$, where v_s has degree $2(p^s - 1)$. From [**13**, §4] we have that δ^*: $K(s)^*(\mathbf{C}P^\infty_+) \to K(s)^*(B(\mathbf{Z}/p^i)_+)$ can be identified with the projection map $K(s)^*[[x]] \to K(s)^*[x]/(x^{p^{is}})$. Combining this with the Kunneth isomorphism,

$$K(s)^*(X) \hat{\otimes}_{K(s)^*} K(s)^*(Y) \simeq K(s)^*(X \wedge Y),$$

yields the algebra structure of $K(s)^*(BP_+)$ and that δ^* is epic in the diagram:

$$\begin{array}{ccc} K(s)^*((\mathbf{C}P^\infty)^n_+) & \overset{A_1^*}{\to} & K(s)^*((\mathbf{C}P^\infty)^n_+) \\ \downarrow \delta^* & & \downarrow \delta^* \\ K(s)^*(BP_+) & \overset{\alpha^*}{\to} & K(s)^*(BP_+) \end{array}$$

Proposition 2.1 now follows from Lemma 2.3 together with the observation that $\mathrm{Tor}(\alpha, \mathbf{Z}/p) = A_1 \otimes \mathbf{Z}/p$: $(\mathbf{Z}/p)^n \to (\mathbf{Z}/p)^n$.

3. Proof of Theorem 1.1. Our proof consists first of a number of reductions.

Reduction to the case $P = (\mathbf{Z}/p^r)^n$. We need the following theorem from [**5**].

THEOREM 3.1. *Let $P = \prod_{i=1}^r (\mathbf{Z}/p^i)^{n_i}$. Any idempotent e in $\mathbf{Z}/p[\mathrm{End}(P)]$ is conjugate to one of the form $e_i \otimes \cdots \otimes e_r$ where e_i is an idempotent in $\mathbf{Z}/p[M_{n_i}(\mathbf{Z}/p^i)]$.*

With notation as in this last theorem, we have

$$\text{(3.2a)} \qquad e\mathbf{Z}/p[P^s] \simeq \bigotimes_{i=1}^{r} e_i \mathbf{Z}/p\left[(\mathbf{Z}/p^i)^{sn_i}\right]$$

and, by the Kunneth isomorphism for $K(s)^*$,

(3.2b) $$k_s(eBP_+) = \prod_{i=1}^{r} k_s\left(e_i B(\mathbf{Z}/p^i)_+^{n_i}\right).$$

It follows that Theorem 1.1 for all P follows from the theorem for P of the form $(\mathbf{Z}/p^r)^n$.

Reduction to the case $P = (\mathbf{Z}/p)^n$. The filtration $(\mathbf{Z}/p^r)^n \supset (\mathbf{Z}/p^{r-1})^n \supset \cdots \supset (\mathbf{Z}/p)^n$ is preserved by $M_n(\mathbf{Z}/p^r)$. This induces a filtration of $\mathbf{Z}/p[(\mathbf{Z}/p^r)^{ns}]$, and the associated graded module is isomorphic to $\mathbf{Z}/p[(\mathbf{Z}/p)^{nrs}]$ where $M_n(\mathbf{Z}/p^r)$ acts on $(\mathbf{Z}/p)^n$ via mod p reduction.

Let $e \in \mathbf{Z}/p[M_n(\mathbf{Z}/p^r)]$ be an idempotent and let $\bar{e} \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$ be its mod p reduction. The above comments imply

(3.3a) $$e\mathbf{Z}/p\left[(\mathbf{Z}/p^r)^{ns}\right] \simeq \bar{e}\mathbf{Z}/p\left[(\mathbf{Z}/p)^{nrs}\right].$$

An inspection of Corollary 2.2 yields

(3.3b) $$k_s\left(eB(\mathbf{Z}/p^r)_+^n\right) = k_{rs}\left(\bar{e}B(\mathbf{Z}/p)_+^n\right).$$

REMARK 3.4. In [5] it is shown that $H^*(eB(\mathbf{Z}/p^r)_+^n) \simeq H^*(\bar{e}B(\mathbf{Z}/p)_+^n)$ as graded vector spaces, and that the multiplicity of $eB(\mathbf{Z}/p^r)_+^n$ in $B(\mathbf{Z}/p^r)_+^n$ equals that of $\bar{e}B(\mathbf{Z}/p)_+^n$ in $B(\mathbf{Z}/p)_+^n$.

Reduction to the case $s = 1$. Let $V = (\mathbf{Z}/p)^n$ and let $e \in \mathbf{Z}/p[\mathrm{End}(V)]$ be an idempotent. Since $\mathbf{Z}/p[V^s] \simeq \mathbf{Z}/p[V]^{\otimes s}$ naturally, we have

(3.5a) $$e\mathbf{Z}/p[V^s] \simeq e\mathbf{Z}/p[V]^{\otimes s}.$$

With notation as in §2, we claim that

(3.5b) $$S_{n,s}e \simeq \left(S_{n,1}^{\otimes s}\right)e.$$

(3.5b) will follow from the next lemma.

LEMMA 3.6. *$S_{n,s}$ can be filtered by right $M_n(\mathbf{Z}/p)$-submodules so that the associated graded module is isomorphic to $S_{n,1}^{\otimes s}$.*

PROOF. We use the notation: $S_{n,s}(V^*) = S_{n,s}$ and ξ: $S_{n,s}(V^*) \to S_{n,s}(V^*)$ is the pth power map. Then $S_{n,s}(V^*)$ is filtered by the subalgebras $A_i \equiv S_{n,s-i}(\xi^i(V^*))$. Then, for $i = 1, \ldots, s$,

$$A_{i-1}//A_i \simeq S_{n,1}(V^*)$$

naturally, so that

$$E_0\left(S_{n,s}(V^*)\right) \simeq S_{n,1}(V^*)^{\otimes s}$$

as $\mathrm{End}(V)$-modules.

Finally we are left needing to prove Theorem 1.1 in the special case when $P = (\mathbf{Z}/p)^n$ and $s = 1$. More precisely, we need to show that

$$\dim e\mathbf{Z}/p\left[(\mathbf{Z}/p)^n\right] = \dim S_{n,1}e$$

for all idempotents $e \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$.

View $V = (\mathbf{Z}/p)^n$ as a restricted Lie algebra with trivial bracket and pth power map. Let $U(V)$ denote the universal enveloping algebra.

PROPOSITION 3.7. *Let $E_0(\mathbf{Z}/p[V])$ be the graded algebra associated to the filtration of $\mathbf{Z}/p[V]$ by the augmentation ideal. Then there is a natural isomorphism of algebras*

$$U(V) \simeq E_0(\mathbf{Z}/p[V]).$$

This is a special case of a more general theorem provided by D. Quillen in **[11]**. (For completeness, we give an explicit proof below.)

Assuming this proposition, we are nearly done. Note that $S_{n,1} \simeq U(V^*) \simeq U(V)^*$ as right $\mathrm{End}(V)$-modules. Thus Proposition 3.7 implies that

$$(e\mathbf{Z}/p[V])^* \simeq S_{n,1}e$$

for all idempotents $e \in \mathbf{Z}/p[\mathrm{End}(V)]$. The proof of Theorem 1.1 is complete.

PROOF OF PROPOSITION 3.7. $U(V)$ is isomorphic to $\mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ with the standard *left* $M_n(\mathbf{Z}/p)$ action. Let $\mathbf{e}_i \in V$ be the ith standard basis vector, and let $\Theta(x_i) = \mathbf{e}_i - \mathbf{0} \in \mathbf{Z}/p[V]$.

LEMMA 3.8. *Θ extends to an isomorphism of algebras*

$$\Theta\colon \mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \simeq \mathbf{Z}/p[V].$$

PROOF. To show that Θ does extend to an algebra map, it suffices to check that $\Theta(x_i)^p = 0$. This is okay: $(\mathbf{e}_i - \mathbf{0})^p = \mathbf{e}_i^p - \mathbf{0}^p = \mathbf{0} - \mathbf{0} = 0$ in $\mathbf{Z}/p[V]$. By dimension counting, to show that Θ is an isomorphism, it suffices to show that Θ is onto. This is easy: $\{\mathbf{e}_i, \ldots, \mathbf{e}_n\}$ is a set of algebra generators for $\mathbf{Z}/p[V]$ and $\mathbf{e}_i = \Theta(1 + x_i)$.

Now note that if $\mathbf{Z}/p[V]$ is filtered by the augmentation ideal, and $\mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ is filtered by degree, then Θ is a filtration-preserving map (i.e., $\Theta(x_i)$ is in the augmentation ideal). The proof of Proposition 3.7 is completed with

LEMMA 3.9. *Θ is an $M_n(\mathbf{Z}/p)$-module map, up to filtration. More precisely, $\Theta(Ax) \equiv A\Theta(x)$ modulo terms of higher filtration, for all $A \in M_n(\mathbf{Z}/p)$, $x \in \mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.*

PROOF. Since $M_n(\mathbf{Z}/p)$ acts on both $\mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ and $\mathbf{Z}/p[V]$ via algebra maps, it suffices to check the result when $x = x_i$. Furthermore, it suffices to assume that A is an "elementary" matrix, i.e. we can assume that A is diagonal, a permutation, or the matrix

$$\begin{pmatrix} 1 & 0 & \\ 1 & 1 & \\ & & I_{n-2} \end{pmatrix}.$$

If A is diagonal or a permutation, then $\Theta(Ax_i) = A\Theta(x_i)$. To check the last possibility it suffices to assume that $n = 2$ (so we are considering $\mathbf{Z}/p[x, y]/(x^p, y^p)$), $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $x_i = x$. In this case, straightforward calculation shows that $A\theta(x) - \Theta(Ax) = \Theta(xy)$, which is in higher filtration.

4. K-theory. In this section we use Theorem 1.1 to prove Theorem 1.5, i.e. we compute $k_1(X)$ where X is an idecomposable wedge summand of $B(\mathbf{Z}/p)^n$. By **[5]**, such spectra are in one-to-one correspondence with the conjugacy classes of primitive idempotents $e \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$. These, in turn, correspond to the p^n distinct (absolutely) irreducible $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-modules, so that, if e corresponds to S, then

$$\dim\ Me = \text{multiplicity of } S \text{ in } M$$

for any $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-module M.

Recall that $S_{n,1} = \mathbf{Z}/p[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$. By the results in §3, $S_{n,1}$ and $(\mathbf{Z}/p[(\mathbf{Z}/p)^n])^*$ have the same irreducible composition factors as right $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-modules. We need to analyze these factors.

Let $S_{n,1}(d)$ be the homogeneous elements in $S_{n,1}$ of degree d. Theorem 1.5 follows from

PROPOSITION 4.1. *The $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-modules $S_{n,1}(d)$, $d = 0, \ldots, (p-1)n$, are all distinct and irreducible.*

We first show that the $S_{n,1}(d)$ are all distinct. For that, we work with $\mathbf{Z}/p[(\mathbf{Z}/p)^n]$.

LEMMA 4.2. *There exists an orthogonal idempotent decomposition $1 = \Sigma_{i=1}^{p^n} e_i \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$ such that $\dim e_i\mathbf{Z}/p[(\mathbf{Z}/p)^n] = 1$ for all i.*

PROOF. Let $D_n \subset M_n(\mathbf{Z}/p)$ be the subsemigroup of diagonal matrices. The idempotent decomposition takes place in $\mathbf{Z}/p[D_n]$. For $n = 1$, the proposition (and thus the lemma) are easy to verify: the $\mathbf{Z}/p[M_1(\mathbf{Z}/p)]$-modules $S_{1,1}(d)$ are irreducible (they are one-dimensional) and distinct (they are powers of the determinant representation). For the general case, note that $\mathbf{Z}/p[D_n] = \mathbf{Z}/p[D_1]^{\otimes n}$, so tensoring the $n = 1$ decomposition yields the lemma.

REMARK 4.3. Geometrically, our argument here corresponds to the following. $B\mathbf{Z}/p_+$ stably decomposes: $B\mathbf{Z}/p_+ \simeq X_0 \vee \cdots \vee X_{p-1}$ with $k_1(X_i) = 1$ ($X_0 = S^0$ and $X_{p-1} = B\Sigma_p$). Thus $B(\mathbf{Z}/p)^n_+$ decomposes into p^n summands,

$$B(\mathbf{Z}/p)^n_+ \simeq \bigvee_{0 \leqslant i_j \leqslant p-1} (X_{i_1} \wedge \cdots \wedge X_{i_n}),$$

each of which has $k_1 = 1$.

Given $I = (i_1, \ldots, i_n)$ with $0 \leqslant i_j \leqslant p-1$, let $e_I \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$ be the idempotent corresponding to $X_{i_1} \wedge \cdots \wedge X_{i_n}$. By Lemma 4.2, in a primitive idempotent decomposition of e_I there will be a unique idempotent ε_I such that $\varepsilon_I\mathbf{Z}/p[(\mathbf{Z}/p)^n] \neq 0$. The number of irreducibles appearing in $\mathbf{Z}/p[(\mathbf{Z}/p)^n]$ will correspond to the number of conjugacy classes of the ε_I.

The group of permutation matrices conjugates the e_I's to others, and thus does the same for the ε_I's. In particular, if $p = 2$, this yields an upper bound of $n + 1$ on the number of conjugacy classes of the ε_I. But a lower bound is given by the number of $S_{n,1}(d)$: again $n + 1$. Proposition 4.1 has thus been proved in this case.

EXAMPLE 4.4. $B\mathbf{Z}/3 \simeq X \vee B\Sigma_3$, stably and localized at 3. Then $B\Sigma_3$ is a wedge summand in $X \wedge X$ (see **[5**, §7**]**).

This example illustrates, geometrically, why our argument for $p = 2$ fails for larger primes. (The example shows that $\varepsilon_{(0,2)}$ is conjugate to $\varepsilon_{(1,1)}$, even though $e_{(0,2)}$ is not conjugate to $e_{(1,1)}$.) Instead, we prove directly that the modules $S_{n,1}(d)$ are irreducible. The author learned of this proof from Dave Carlisle and Reg Wood.

For $I = (i_1, \ldots, i_n)$ with $0 \leqslant i_j \leqslant p - 1$, let $|I| = i_1 + \cdots + i_n$, and let $x^I = x_1^{i_1} \cdots x_n^{i_n} \in S_{n,1}$. The irreducibility of $S_{n,1}(d)$ follows from a characteristic p version of Lemma 2.4 of [**3**].

LEMMA 4.5. *Given I, J with $|I| = |J| = d$, there exists $\Theta_{I,J} \in \mathbf{Z}/p[M_n(\mathbf{Z}/p)]$ such that*

$$\Theta_{I,J}(x^K) = \begin{cases} x^J & \text{if } K = I, \\ 0 & \text{if } K \neq I, |K| = d. \end{cases}$$

PROOF. The existence of $\Theta_{I,I}$ follows exactly as in [**3**]. (In fact, $\Theta_{I,I} \in \mathbf{Z}/p[D_n]$.) Armed with the $\Theta_{I,I}$, it suffices to show that there exists $\Psi_{I,J}$ such that

$$\Psi_{I,J}(x^I) = x^J + \text{other terms},$$

since we can then let $\Theta_{I,J} = \Theta_{J,J} \circ \Psi_{I,J} \circ \Theta_{I,I}$. To show the existence of $\Psi_{I,J}$ it suffices to assume that $n = 2$, $I = (i + 1, j)$, and $J = (i, j + 1)$. This is then easy to verify:

$$(x + ay)^{i+1} y^j = a(i + 1)x^i y^{j+1} + \text{other terms},$$

so letting $a = (i + 1)^{-1} \in (\mathbf{Z}/p)^*$ yields a linear substitution $\Psi_{I,J}$.

5. The k_s-sequence. In this section, we study the sequence of numbers $k_0(X)$, $k_1(X)$, $k_2(X), \ldots$ where X is a wedge summand of BP_+, and prove Theorem 1.6. By the reductions (3.2b) and (3.3b) of §3, it suffices to prove Theorem 1.6 in the case when $P = (\mathbf{Z}/P)^n$, except that statement (2) of the theorem needs to be strengthened to

(2′) If X is a summand of $B(\mathbf{Z}/p)^n_+$ then, for any $r = 1, 2, \ldots$,

$\sum_{s=0}^{\infty} k_{rs}(X)t^s$ is a rational function with poles contained in the set $\{p^{-s} \mid s = 0, 1, 2, \ldots\}$.

Using Theorem 1.1, we need to study the sequence

$$\{\dim e\mathbf{Z}/p[M_{n,s}(\mathbf{Z}/p)] \mid s = 0, 1, 2, \ldots\},$$

where e is an idempotent in $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$. We examine the structure of $M_{n,s}(\mathbf{Z}/p)$ as a left $M_n(\mathbf{Z}/p)$-set.

If V is a subspace of $(\mathbf{Z}/p)^s$, let $M_V \subset M_{n,s}(\mathbf{Z}/p)$ be the set of all matrices with rows which are vectors in V.

LEMMA 5.1. *The M_V satisfy the following properties*:

(1) *M_V is a left $M_n(\mathbf{Z}/p)$-set.*

(2) *M_V is filtered by rank as a left $M_n(\mathbf{Z}/p)$-set.*

(3) *If V and V' are isomorphic subspaces of $(\mathbf{Z}/p)^s$, then $M_V \simeq M_{V'}$ as filtered left $M_n(\mathbf{Z}/p)$-sets.*

(4) $M_V \cap M_{V'} = M_{V \cap V'}$.

DEFINITIONS 5.2. (1) Let $a_{s,k}$ = number of k-planes in $(\mathbf{Z}/p)^s$

$$= \frac{(p^s - 1)\cdots(p^{s-k+1} - 1)}{(p^k - 1)\cdots(p-1)}.$$

(2) For $0 \leqslant k \leqslant n$, let $M'_{n,k} \subset M_{n,k}(\mathbf{Z}/p)$ be the set of matrices of rank $< k$, and let $N_{n,k}$ be the $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-module $\mathbf{Z}/p[M_{n,k}(\mathbf{Z}/p)]/\mathbf{Z}/p[M'_{n,k}]$.

With these definitions, Lemma 5.1 has the following corollary.

COROLLARY 5.3. *As left $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$-modules,*

$$\mathbf{Z}/p[M_{n,s}(\mathbf{Z}/p)] \quad \textit{and} \quad \bigoplus_{k=0}^{n} a_{s,k} N_{n,k}$$

have the same irreducible composition factors.

Theorem 1.6 will now follow from an examination of the coefficients $a_{s,k}$. Let A_n be the $(n+1)\times(n+1)$ matrix with (s,k) entry $a_{s,k}$. Note that A_n is lower triangular, and thus invertible. The linear function $\alpha_{n,s}$: $\mathbf{Z}^{n+1} \to \mathbf{Z}$ of Theorem 1.6(1) will be the composite

$$\mathbf{Z}^{n+1} \xrightarrow{A_n^{-1}} \mathbf{Z}^{n+1} \xrightarrow{(a_{s,0},\ldots,a_{s,n})} \mathbf{Z}. \tag{5.4}$$

To prove statements (2′) and (3) we use the following observation:

LEMMA 5.5. *Let $\alpha_0, \alpha_1, \ldots$ be a sequence of elements in* $\mathrm{Hom}(\mathbf{Z}^m, \mathbf{Z})$. *Let $v_1, \ldots, v_m$ form a basis for $\mathbf{Z}^m$.*

(1) *Suppose the sequence $\alpha_0(v)$, $\alpha_1(v), \ldots$ converges, in the p-adic topology, to an element in $\mathbf{Z}_{(p)}$ for $v = v_1, \ldots, v_m$. Then the same is true for all $v \in \mathbf{Z}^m$.*

(2) *Suppose $f_v(t) = \sum_{s=0}^{\infty} \alpha_s(v) t^s$ is a rational function with poles contained in the set $\{p^{-s} \mid s = 0, 1, 2, \ldots\}$ for $v = v_1, \ldots, v_m$. Then the same is true for all $v \in \mathbf{Z}^m$.*

Applying this lemma to the basis $\{A_n e_0, \ldots, A_n e_n\}$ where $\{e_0, \ldots, e_n\}$ is the standard basis of $\mathbf{Z}^{n+1}$, we see that statements (2′) and (3) of Theorem 1.6 follow from the next two lemmas.

LEMMA 5.6. *For all r, k, the function $\sum_{s=0}^{\infty} a_{rs,k} t^s$ is a rational function with poles contained in the set $\{\bar{p}^s \mid s = 0, 1, 2, \ldots\}$.*

LEMMA 5.7. *For all k, the sequence $a_{0,k}, a_{1,k}, a_{2,k}, \ldots$ converges, in the p-adic topology, to an element of $\mathbf{Z}_{(p)}$.*

PROOF OF LEMMA 5.6.

$$\sum_{s=0}^{\infty} a_{r,s} t^s = \frac{1}{(p^k - 1)\cdots(p-1)} \sum_{s=0}^{\infty} (p^{rs} - 1)\cdots(p^{rs-k+1} - 1) t^s$$

which is a linear combination of functions of the form

$$\sum_{s=0}^{\infty} p^{as} t^s = \frac{1}{(1 - p^a t)}.$$

PROOF OF LEMMA 5.7. In the p-adic topology,

$$\lim_{s\to\infty} a_{s,k} = \lim_{s\to\infty} \frac{(p^s-1)\cdots(p^{s-k+1}-1)}{(p^k-1)\cdots(p-1)} = \frac{1}{(1-p^k)\cdots(1-p)}.$$

REMARK 5.8. There is a recursion relation, easily verified,

$$a_{s+1,k} - a_{s,k} = p^{s-k+1}a_{s,k-1}.$$

This implies, for example, that p^{s-n+1} divides $(k_{s+1}(X) - k_s(X))$ where X is a wedge summand of $B(\mathbf{Z}/p)^n$. This can be improved slightly: noting that $N_{n,n} \simeq \mathbf{Z}/p[\mathrm{GL}_n(\mathbf{Z}/p)]$, where the projection $\mathbf{Z}/p[M_n(\mathbf{Z}/p)] \to \mathbf{Z}/p[\mathrm{GL}_n(\mathbf{Z}/p)]$ defines the module structure, we see that $p^{\binom{n}{2}}$ (= order of the p-Sylow subgroup of $\mathrm{GL}_n(\mathbf{Z}/p)$) divides $\dim eN_{n,n}$ for all idempotents $e \in \mathbf{Z}[M_n(\mathbf{Z}/p)]$. Thus we have that if $n \geqslant 2$, p^{s-n+2} divides $(k_{s+1}(X) - k_s(X))$, with X as above.

6. Further remarks and examples. As a practical matter, Theorem 1.2 is easiest to use when W acts freely on $P - \{0\}$. In this case,

$$k_s(G) = \left[(|P|^s - 1)/|W|\right] + 1.$$

EXAMPLES 6.1. (1) $G = \mathrm{GL}_2(\mathbf{F}_q)$, $q = p^n$. P is the unipotent subgroup of upper triangular matrices with 1's on the diagonal, $C_G(P)$ includes the constant diagonal matrices, and $N_G(P)$ is the set of all upper triangular matrices. It is easy to check that $W \simeq \mathbf{F}_q^*$ acting in the usual way on $P \simeq \mathbf{F}_q$. Thus

$$k_s(\mathrm{GL}_2(\mathbf{F}_q)) = (q^s-1)/(q-1) + 1.$$

(2) $G = \mathrm{SL}_2(\mathbf{F}_q)$ *or* $\mathrm{PSL}_2(\mathbf{F}_q)$, $q > 2$. This is as in (1), except that now $W \simeq$ squares in $\mathbf{F}_q^*$. Thus

$$k_s(\mathrm{SL}_2(\mathbf{F}_q)) = k_s(\mathrm{PSL}_2(\mathbf{F}_q)) = 2(q^s-1)/(q-1) + 1.$$

EXAMPLE 6.2. With $p = 3$, let W be a 2-Sylow subgroup of $\mathrm{GL}_2(\mathbf{Z}/3)$ (of order 16). Let $G = (\mathbf{Z}/3)^2 \rtimes W$. Then $P = (\mathbf{Z}/3)^2$, and W has four distinct subgroups of order 2 each of which is the isotropy subgroup of two distinct elements of $(\mathbf{Z}/3)^2$. Counting W-orbits in P^s having various isotropy subgroups leads to

$$k_s(G) = \left[(3^s+2)^2 + 7\right]/16.$$

The next examples use the methods of §5.

EXAMPLE 6.3. If X is one of the $(p-1)$ indecomposable summands of $B\mathbf{Z}/p$, then $k_s(X) = (p^s-1)/(p-1)$.

EXAMPLE 6.4. As in [**5**, §7], we use the notation $X_{i,j}$, $0 \leqslant i, j \leqslant p-1$, to denote the p^2 distinct irreducible summands of $B(\mathbf{Z}/p)^2_+$, and $S_{i,j}$ to denote the corresponding irreducible $\mathbf{Z}/p[M_2(\mathbf{Z}/p)]$-module. Here $X_{i,0} = X_i$ of Remark 4.3 (i.e. a summand of $B\mathbf{Z}/p^+$), and $S_{i,j} = S_{i,0} \otimes (\det)^j$. $X_{0,0} = S^0$. In [**4**], D. J. Glover computed the dimensions of the projective covers of the $S_{i,j}$. This amounts to computing $k_2(X_{i,j})$ for all i and j, and, by Remark 5.8, $k_1(X_{i,j})$ can also be immediately computed. We read off the following table.

TABLE 1

(i, j)		$k_1(X_{i,j})$	$k_2(X_{i,j})$
$(i, 0)$	$0 < i \leqslant p - 1$	1	$p + 1$
$(0, p - 1)$		1	$2p + 1$
(i, j)	$i + j = p - 1$ and $i, j > 0$	1	$3p + 1$
$(0, j)$	$0 < j < p - 1$	0	p
$(p - 1, j)$	$0 < j \leqslant p - 1$	0	p
(i, j)	all other $(i, j) \neq (0, 0)$	0	$2p$

By (5.4), $k_s(X_{i,j}) = a_{s,2}k_2(X_{i,j}) + a_{s,1}[1 - (p^{s-1} - 1)/(p - 1)]k_1(X_{i,j})$ for $(i, j) \neq (0, 0)$.

EXAMPLE 6.5. Let $L(n) = \Sigma^{-n}SP^{p^n}(S)/SP^{p^{n-1}}(S)$ [9]. $L(n)$ is an indecomposable summand of $B(\mathbf{Z}/p)^n$ and corresponds to the Steinberg representation of $\mathbf{Z}/p[\mathrm{GL}_n(\mathbf{Z}/p)]$ pulled back to $\mathbf{Z}/p[M_n(\mathbf{Z}/p)]$. We first note that $k_s(L(n)) = 0$ for $s < n$. Welcher showed this topologically in [**14**] (see also [**10**, §4]). For an algebraic proof, it suffices to show that $e_n N_{n,k} = 0$ for $k < n - 1$, where $e_n \in \mathbf{Z}/p[\mathrm{GL}_n(\mathbf{Z}/p)]$ is a Steinberg idempotent, since $e_n B(\mathbf{Z}/p)^n_+ \simeq L(n) \vee L(n - 1)$. The relevant computation is easy and appears in [**8**]. With this information, it follows that

$$k_n(L(n)) = \dim e_n N_{n,n} = p^{\binom{n}{2}}.$$

We conclude

$$k_s(L(n)) = a_{s,n} p^{\binom{n}{2}}.$$

REMARK 6.6. With this formula, one can check that, for all s,

$$\sum_{n=0}^{s} (-1)^s k_s(L(n)) = 0.$$

The author has recently discovered [**15**] that formulae like this occur whenever $K(s)^*(\ \ )$ is applied to a "spacelike" resolution of a spectrum (e.g. the $L(n)$ sequence of [**9**]).

REMARK 6.7. Suppose that X is an indecomposable summand of $B(\mathbf{Z}/p)^n$ such that $k_s(X) = 0$ for $s < n$ (e.g. $X = L(n)$). Call such a summand *regular*. As in the last example, it follows that if X is regular, then $k_s(X) = a_{s,n}k_n(X)$, and $k_n(X)$ will be the dimension of an indecomposable projective $\mathbf{Z}/p[\mathrm{GL}_n(\mathbf{Z}/p)]$-module. We conjecture that almost all the indecomposable summands of $B(\mathbf{Z}/p)^n$ are regular. More precisely, we conjecture that, with $r(n, p) =$ number of regular summands of $B(\mathbf{Z}/p)^n$, $\lim_{p \to \infty} r(n, p)/p^n = 1$. This is true for $n \leqslant 2$.

REFERENCES

1. J. F. Adams, J. H. Gunawardena and H. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), 435–460.

2. M. F. Atiyah, *Characters and cohomology of finite groups*, Publ. Math. Inst. Hautes Etudes Sci. **9** (1961), 247–288.

3. D. Carlisle, P. Eccles, S. Hilditch, N. Ray, L. Schwartz, G. Walker and R. Wood, *Modular representations of* GL(n, p), *splitting* $\Sigma(\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty)$, *and the β-family as framed hypersurfaces*, Math. Z. **189** (1985), 239–261.

4. D. J. Glover, *A study of certain modular representations*, J. Algebra **51** (1978), 425–475.

5. J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p-groups*, Math. Proc. Cambridge Philos. Soc. (submitted).

6. M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory*. II (in preparation).

7. N. J. Kuhn, *The* mod p *K-theory of classifying spaces of finite groups*, J. Pure Appl. Algebra **44** (1987), 269–271.

8. N. J. Kuhn, *The rigidity of* $L(n)$, Proc. Topology year at Univ. of Washington (to appear).

9. N. J. Kuhn and S. B. Priddy, *The transfer and Whitehead's conjecture*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 459–480.

10. S. A. Mitchell, *Finite complexes with* $A(n)$*-free cohomology*, Topology **24** (1985), 227–248.

11. D. G. Quillen, *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418.

12. D. C. Ravenel, *Morava K-theories and finite groups*, Symposium on Algebraic Topology in Honor of José Adem, Contemp. Math., vol. 12, Amer. Math. Soc., Providence, R. I., 1982, pp. 289–292.

13. D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture*, Amer. J. Math. **102** (1980), 691–748.

14. P. J. Welcher, *Symmetric fiber spectra and* $K(n)$*-homology acyclicity*, Indiana J. Math. **39** (1981), 801–812.

15. N. J. Kuhn, *Morova K-theories and infinite loop spaces*, Proc. 1986 Arcata Topology Conference (to appear).

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

Current address: Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903